Graphic Representation of the Impedance of Networks Containing Resistances and Two Reactances

By CHARLES W. CARTER, Jr.

ABSTRACT: The driving-point impedance of an electrical network composed of any number of resistances, arranged in any way, and two pure reactances, of any degree of complication within themselves but not related to each other by mutual reactance, inserted at any two points in the resistance network, is limited to an eccentric annular region in the complex plane which is determined by the resistance network alone.

The boundaries of this region are non-intersecting circles centered on the axis of reals. The diameter of the exterior boundary extends from the value of the impedance when both reactances are short-circuited to its value when both are open-circuited. The diameter of the interior boundary extends from the value of the impedance when one reactance is short-circuited and the other open-circuited to its value when the first reactance is open-circuited and the second short-circuited.

When either reactance is fixed and the other varies over its complete range, the locus of the driving-point impedance is a circle tangent to both boundaries. By means of this grid of intersecting circles the locus of the driving-point impedance may be shown over any frequency range or over any variation of elements of the reactances. This is most conveniently done on a doubly-sheeted surface.

The paper is illustrated by numerical examples.

INTRODUCTION

SUPPOSE that any number of resistances are combined into a network of any sort and provided with three pairs of terminals, numbered (1) to (3) as in Fig. 1. The problem set in this paper is to investigate the driving-point impedance[1] of such a network at

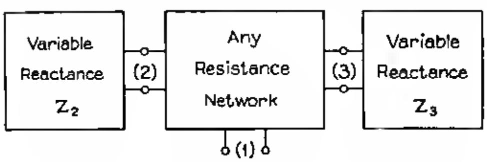


Fig. 1—The Network to be Discussed

terminals (1) when variable pure reactances, Z_2 and Z_3, are connected to terminals (2) and (3), respectively. Z_2 and Z_3 are formed of capacities, self and mutual inductances. They are not connected to each other by mutual reactance, but they may be of any degree of complication within themselves.

The problem is dealt with in terms of the complex plane: that is, the resistance components of the impedance, S, measured at terminals

[1] The driving-point impedance of a network is the ratio of an impressed electromotive force at a point in a branch of the network to the resulting current at the same point.

(1) are plotted as abscissas and the reactance components as ordinates. To every value of the impedance, then, there is a corresponding point, and to the values of the impedance over a range of variation of some element, or over a frequency range, there corresponds a locus, in the complex plane. This locus may be labelled at suitable points with the corresponding value of the variable. So labelled, it combines into one the curves which are usually plotted to show separately the variation of the reactance and resistance components or to show separately the variation of absolute value and angle.

The use of the complex plane is not new: it is the basis of most of the vector diagrams for electrical machinery. The characteristics of both smooth and loaded transmission lines have also been displayed by its means. Its application to electrical networks, however, is not common, and it is a subsidiary purpose of this paper to illustrate the fact that the properties of certain networks, which have complicated characteristics if exhibited in the usual way, may be shown quite simply in the complex plane. This simplicity, combined with generality, is attained by application of theorems concerning functions of a complex variable which are immediately available.

The Fundamental Equations

The impedance measured in branch 1 of any network is

$$S=R+iX=\frac{\Delta}{\Delta_{11}} \tag{1}$$

where Δ is the discriminant of the network, either in terms of branches or n independent meshes.[2]

Assigning the reactances Z_2 and Z_3 to meshes 2 and 3

$$\Delta=\begin{vmatrix} R_{11} & R_{12} & R_{13} & . & . & R_{1n} \\ R_{21} & R_{22}+Z_2 & R_{23} & . & . & R_{2n} \\ R_{31} & R_{32} & R_{33}+Z_3 & . & . & R_{3n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ R_{n1} & R_{n2} & R_{n3} & . & . & R_{nn} \end{vmatrix} \tag{2}$$

where R_{jj} is the resistance in mesh j and $R_{jk}(=R_{kj})$ that common to meshes j and k.

[2] See: G. A. Campbell, Transactions of the A. I. E. E., 30, 1911, pages 873–909, for a complete discussion of the solution of networks by means of determinants.

Therefore
$$S=\frac{A+A_{22}Z_2+A_{33}Z_3+A_{22\cdot 33}Z_2Z_3}{A_{11}+A_{11\cdot 22}Z_2+A_{11\cdot 33}Z_3+A_{11\cdot 22\cdot 33}Z_2Z_3} \tag{3}$$

where A is the discriminant of the resistance network alone and $A_{jj\cdot kk\cdot ll}$ denotes the cofactor of the product of the elements of A located at the intersections of rows j, k and l with columns j, k and l, respectively.

For convenience this is written as

$$S=\frac{a+bZ_2+cZ_3+dZ_2Z_3}{a_1+b_1Z_2+c_1Z_3+d_1Z_2Z_3}. \tag{4}$$

The constants of (3) and (4) are real and positive since they are cofactors of terms in the leading diagonal of the discriminant of a resistance network. The determinant being symmetrical, there is the following relation among them:

$$(ad_1-a_1d+bc_1-b_1c)^2=4(bd_1-b_1d)(ac_1-a_1c). \tag{5}$$

The function to be studied is, then, a rational function of two variables, having positive real coefficients determined by the resistances alone. Furthermore, if one reactance is kept constant while the other is varied, the function is bilinear. The particular property of the bilinear function, which has been studied in great detail, of interest here, is that by it circles are transformed into circles.[3]

When, as in this case, the variable in a bilinear function is a pure imaginary, the function may be rewritten in a form which gives directly the analytical data needed. For suppose

$$w=\frac{u+vz}{u_1+v_1z} \tag{6}$$

where z is a pure imaginary and the coefficients are complex. This is

$$w=\frac{v}{v_1}+\frac{u-u_1v/v_1}{u_1+v_1z}. \tag{7}$$

Multiplying the second term by a factor identically unity,

$$w=\frac{v}{v_1}+\frac{u-u_1v/v_1}{u_1+v_1z}\times\frac{v_1'(u_1+v_1z)+v_1(u_1'+v_1'z')}{u_1v_1'+u_1'v_1} \tag{8}$$

where primes indicate conjugates, or

$$w=\frac{uv_1'+u_1'v}{u_1v_1'+u_1'v_1}+\frac{uv_1-u_1v}{u_1v_1'+u_1'v_1}\times\left(\frac{u_1'+v_1'z'}{u_1+v_1z}\right). \tag{9}$$

[3] G. A. Campbell discusses, in the paper cited, the theorem that if a single element of any network be made to traverse any circle whatsoever, the driving-point impedance of the network will also describe a circle.

Now, as z is varied, the first term is constant. In the second term the first factor is constant and the second factor varies only in angle, since the numerator is the conjugate of the denominator. The first term, therefore, is the center, and the absolute value of the first factor of the second term is the radius, of the circle in which w moves as z takes all imaginary values.

One Variable Reactance Giving Circular Locus

The significance of the equations may be made apparent by a study of Fig. 2, which shows the impedance S when one of the reactances, say Z_3, is made zero. We have, then,

$$S=\frac{A+A_{22}Z_2}{A_{11}+A_{11\cdot 22}Z_2}=\frac{a+bZ_2}{a_1+b_1Z_2} \tag{10}$$

and the trivial case $ab_1-a_1b=0$ is excluded. This is of the type of (6). When Z_2 varies over all pure imaginary values, S traces out a

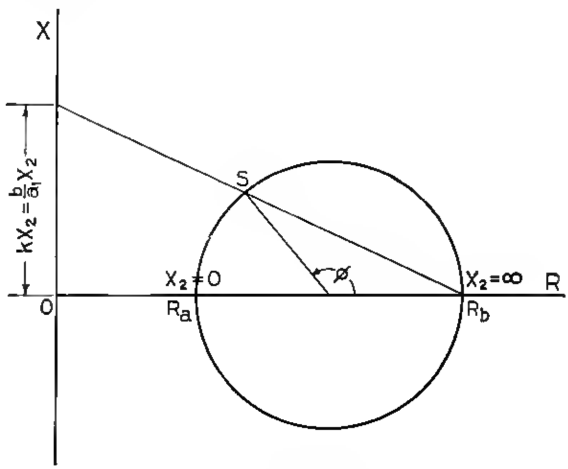


Fig. 2—Locus of the Impedance S with One Variable Reactance

circle, which (9) shows has its center on the resistance axis. Its intercepts on the resistance axis are

$$S=\frac{a}{a_1}=R_a\text{, say, when } Z_2=0 \tag{11}$$

and

$$S=\frac{b}{b_1}=R_b\text{, when } Z_2=\infty\,. \tag{12}$$

But in a symmetrical determinant

$$A_{11}A_{22}-A_{12}^2=AA_{11\cdot 22}; \tag{13}$$

therefore

$$ab_1<a_1b \tag{14}$$

or

$$\frac{a}{a_1}<\frac{b}{b_1} \tag{15}$$

whence

$$R_a<R_b. \tag{16}$$

To find the value of S when Z_2 has some value, say $Z_2=iX_2$, it 's only necessary to mark the circular locus with a scale in terms of Z_2. This may be done directly by using (9) to determine the angle, ϕ, which the radius of the circle makes when $Z_2=iX_2$. It is simpler to use the fact that a line passing through R_b and the point S has an intercept on the reactance axis of

$$X_0=\kappa X_2 \tag{17}$$

where $\kappa=b/a_1$.

The factor κ is determined by the resistances; therefore the scale, as well as the locus, is completely fixed by the resistances. Since κ is always positive, as X_2 is increased the circle is traversed in a clockwise sense; for positive values of X_2 the upper semi-circle is covered; for negative values, the lower. That is, when Z_2 is an inductance the impedance of the network varies on the upper semi-circle from R_a to R_b as the frequency is increased from zero to infinity. When the magnitude of Z_2 is changed the same semi-circle is described but each point (except the initial and final ones) is reached at a different frequency. When Z_2 is a capacity the lower semi-circle, from R_b to R_a, is traced out.

We know that, in general, the value of a pure reactance [4] increases algebraically with frequency, and that its resonant and anti-resonant frequencies alternate, beginning with one or the other at zero frequency. When Z_2 is a general reactance, therefore, as the frequency increases the entire circle is described in a clockwise sense between each consecutive pair of resonant (or anti-resonant) frequencies. For example, if Z_2 is made up of n branches in parallel, one being an inductance, one a capacity and the others inductance in series with capacity, as the frequency increases from zero to infinity the circle is traced out completely $n-1$ times commencing with R_a.

[4] See: A Reactance Theorem, R. M. Foster, *Bell System Technical Journal*, April, 1924, pages 259–267; also: Theory and Design of Uniform and Composite Electric Wave-Filters, O. J. Zobel, *Bell System Technical Journal*, January, 1923, pages 1–47, especially pages 35–37.

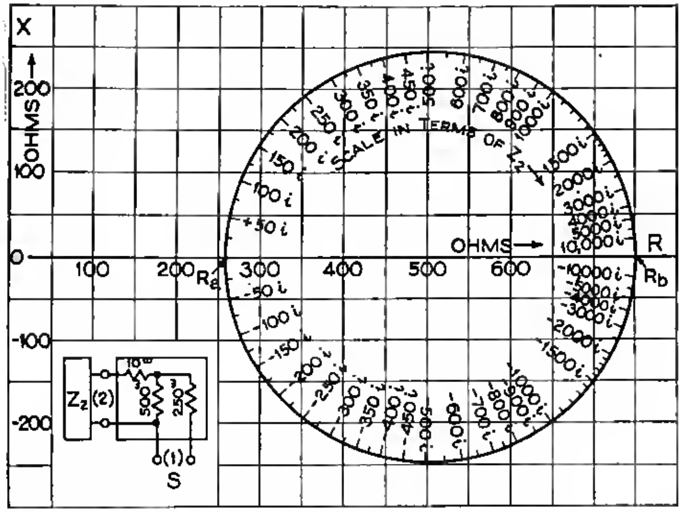


Fig. 3—Impedance of Resistance Network Containing One Variable Reactance

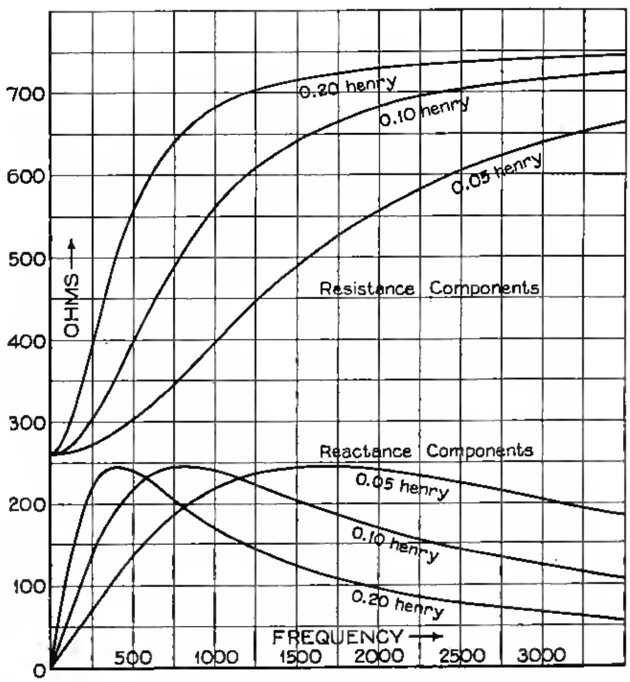


Fig. 3a—Components of Impedance in Fig. 3 when Z_2 is an Inductance Having the Values 0.05, 0.10, and 0.20 Henry

In Fig. 3 is shown the impedance locus for the particular network given on the diagram. The circle is marked in terms of Z_2. From it, certain properties of S may be read at once: the resistance component, R, varies between 260 and 750 ohms, and the reactance

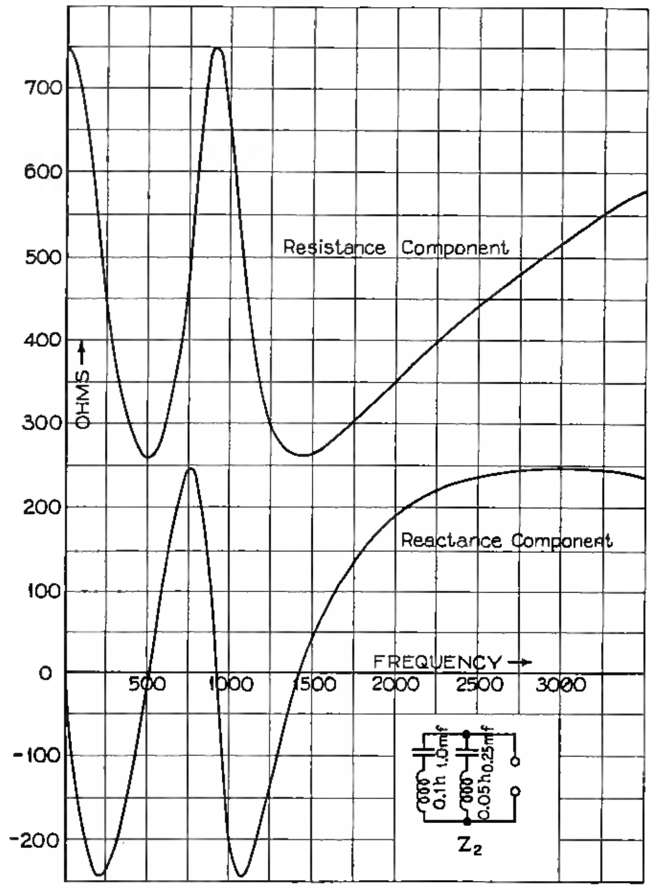


Fig. 3b—Components of Impedance in Fig. 3 when Z_2 is Doubly-Resonant

component, X, is not greater than 245 ohms nor less than -245 ohms, attaining these values when Z_2 is $+510i$ and $-510i$, respectively.

When the variation of the reactance Z_2 with frequency is known the variation of R and X with frequency may be found by using the scale on the circle. For a particular reactance network, the scale may be marked directly in terms of frequency, or if it is desired to compare the behavior of R and X when different reactance networks are sub-

stituted, the impedance locus may be marked with the frequency scale for each reactance network in some distinctive manner.

However, to show in the usual way some of the types of R and X curves represented by the locus of Fig. 3, as well as to avoid needless complication of what is intended as an illustrative rather than a working drawing, Figs. 3a and 3b have been prepared by direct projection from Fig. 3. In Fig. 3a are shown the R and X curves plotted against frequency when Z_2 is an inductance. In Fig. 3b are shown similar curves when Z_2 is a doubly-resonant reactance. The R component has a minimum at each resonant frequency and a maximum at each anti-resonant frequency, while the X component becomes zero at resonant and anti-resonant frequencies alike. The number of examples from this one resistance network might be multiplied endlessly; it is believed, however, that these are sufficient to show the great amount of information to be obtained in very compact form from one simple figure in the complex plane, and the especial superiority of the complex plane in displaying the characteristic common to all the curves of Figs. 3a and 3b: namely, that R and X at any frequency, with any reactance network, are such that the impedance lies on one circle.

Two Variable Reactances Giving Eccentric Annular Domain

Returning to the more general impedance of (4) it is seen that in each case short-circuiting and open-circuiting the terminals (2) and (3) one at a time, and varying the reactance across the other terminals, yields a locus for S which is a circle of the type just discussed. These circles are determined as follows:

Circle	Extremities of Diameter	Scale Factor κ
$Z_2=0$	R_a and R_c	c/a_1
$Z_2=\infty$	R_b and R_d	d/b_1
$Z_3=0$	R_a and R_b	b/a_1
$Z_3=\infty$	R_c and R_d	d/c_1

where $R_c=c/c_1$ and $R_d=d/d_1$. An examination similar to that in (13)–(15) shows that

$$R_a \leq R_b \leq R_d, \tag{18}$$

$$R_a \leq R_c \leq R_d. \tag{19}$$

It may furthermore be assumed without loss of generality, since it is merely a matter of labelling the reactances Z_2 and Z_3, that $R_b \leq R_c$.

Hence, the four critical points of the impedance are always in the following order:

$$R_a \leq R_b \leq R_c \leq R_d. \tag{20}$$

These circles are shown in Fig. 4. By means of the appropriate scale factors κ each may be marked in terms of the reactance which is left in the circuit.

Now suppose Z_3 is kept constant at some value which is a pure imaginary, and Z_2 is varied over the range $-i\infty \leq Z_2 \leq +i\infty$. We may rewrite (4) in the normal form (6):

$$S = \frac{a + cZ_3 + (b + dZ_3)Z_2}{a_1 + c_1Z_3 + (b_1 + d_1Z_3)Z_2}. \tag{21}$$

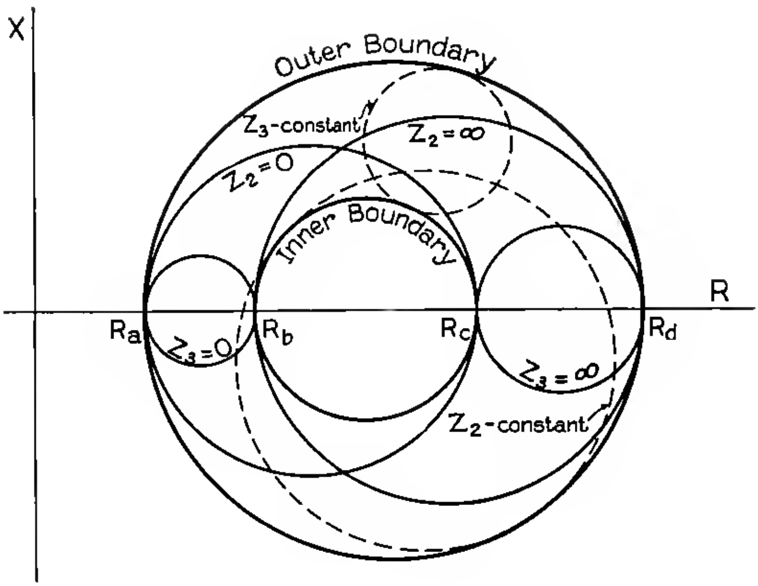


Fig. 4—The Region to which S is Restricted and the Critical Circles

The locus of S is one of a family of circles, each circle corresponding to a value of Z_3 and completely traced out by complete variation of Z_2. The properties of each circle may be found by substitution in (9).

Similarly, if Z_2 is held constant while Z_3 varies, the locus of S is one of another family of circles.

By the use of (9), keeping (5) in mind, it may be shown that the circles of each of these families are tangent to two circles determined by the resistance network alone. Both families are tangent internally to a circle centered on the resistance axis, extending from R_a to R_d. Both are tangent to a circle centered on the resistance axis, extending from R_b to R_c, in such a way that the Z_3-constant circles are tangent *externally* and the Z_2-constant circles are tangent *enclosing* the circle from R_b to R_c. These relationships are illustrated in Fig. 4.

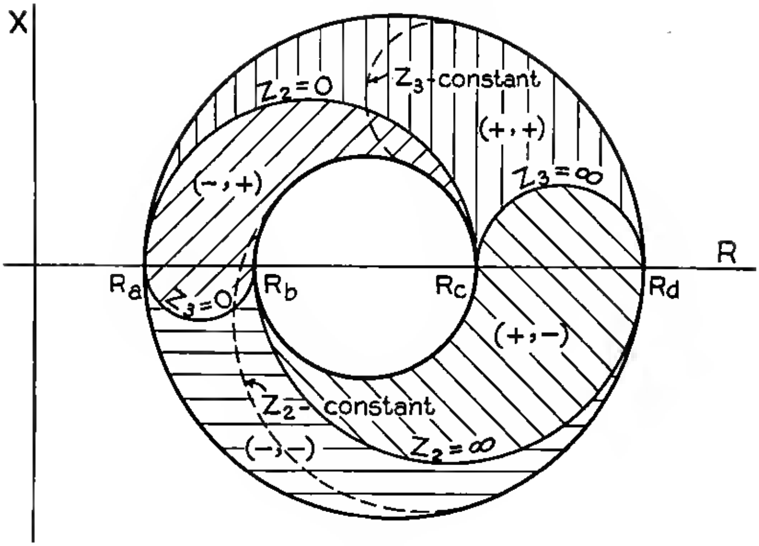
The circles R_a to R_d and R_b to R_c are, therefore, outer and inner boundaries, respectively, of the region mapped out by the two families of circles generated when first one and then the other reactance is treated as a parameter while the remaining reactance is treated as the variable. No matter what reactances may be attached to terminals (2) and (3), the resistance component R, measured at terminals (1), is not greater than the resistance when terminals (2) and (3) are open and not less than the resistance when terminals (2) and (3) are short-circuited, and the reactance component X, measured at terminals (1), is not greater in absolute value than half the difference of the resistances measured when terminals (2) and (3) are open and short-circuited. That is,

$$R_a \leq R \leq R_d, \tag{22}$$

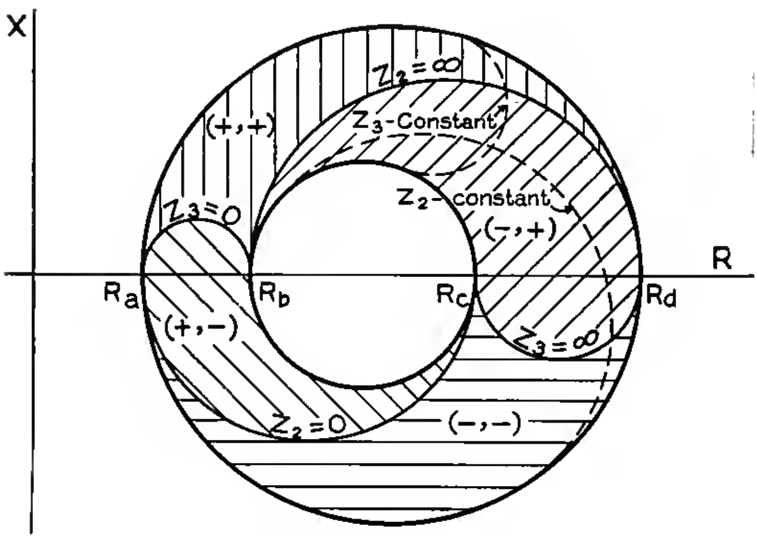
$$|X| \leq \tfrac{1}{2}(R_d - R_a). \tag{23}$$

The two families of circles (Z_2-constant and Z_3-constant) intersect and may be used as a coordinate system from which the components of S may be read for any pair of values Z_2, Z_3. To avoid intersections giving extraneous values of S resort is made to a doubly-sheeted surface, analogous to a Riemann surface, for which the two boundary circles are junction lines. That is, the impedance plane is conceived of as two superposed sheets, transition from one to the other being made at the boundary circles. Thus, in Fig. 5, where the two sheets are separated, each Z_2-constant circle is shown running from the outer to the inner boundary in Sheet I (using the clockwise sense), and from the inner to the outer boundary in Sheet II, while the Z_3-constant circles run from the inner to the outer boundary in Sheet I and are completed in Sheet II.[5]

[5] It may be mentioned that the inner and outer boundaries are impedance curves traced out when $Z_2Z_3 = \dfrac{A_{12}A_{13}}{A_{12\cdot 33}A_{13\cdot 22}}$ and $\dfrac{Z_2}{Z_3} = \dfrac{A_{13}A_{12\cdot 33}}{A_{12}A_{13\cdot 22}}$, respectively.



Sheet I



Sheet II

Fig. 5—The Doubly-Sheeted Surface

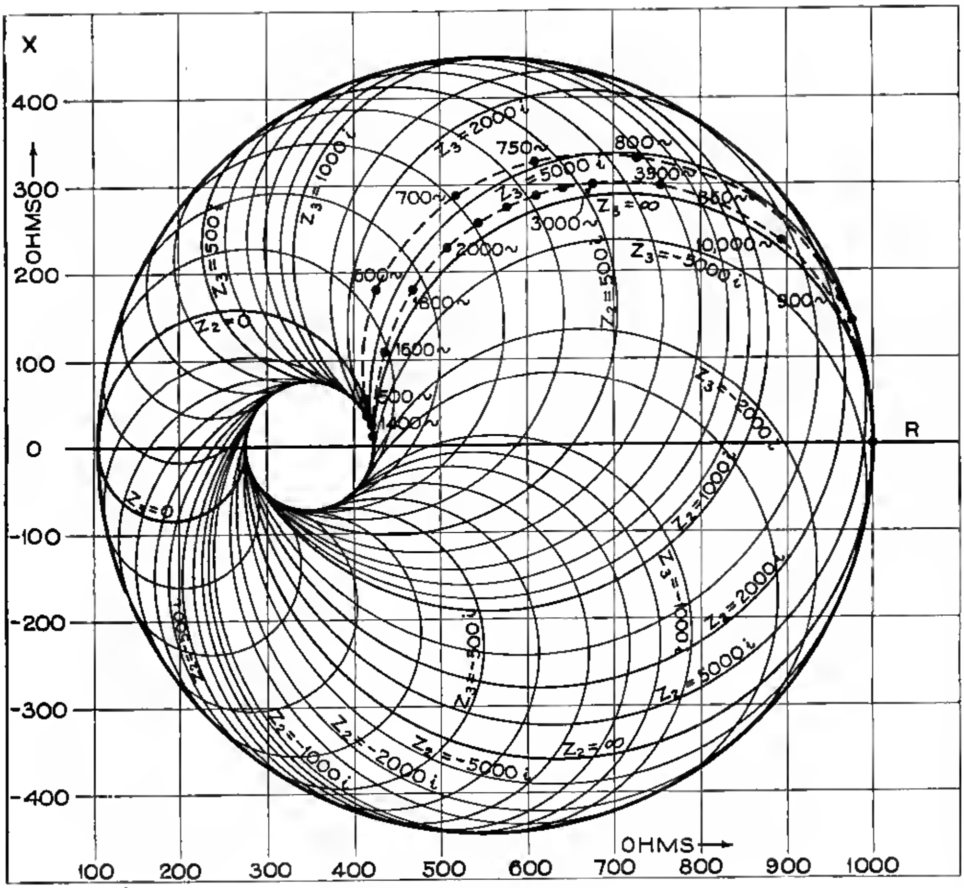


Fig. 6—Sheet I

Sheets I and II of Fig. 6, taken together, show the impedance domain of the network at the bottom of the opposite page, made up of three fixed resistances and two variable reactances. The dashed curve, appearing in four distinct parts, two on each sheet, shows the impedance S when Z_2 is the doubly-resonant circuit of Fig. 3b, and Z_3 is an inductance of 1.0 henry. Points on this curve are labelled in terms of frequency

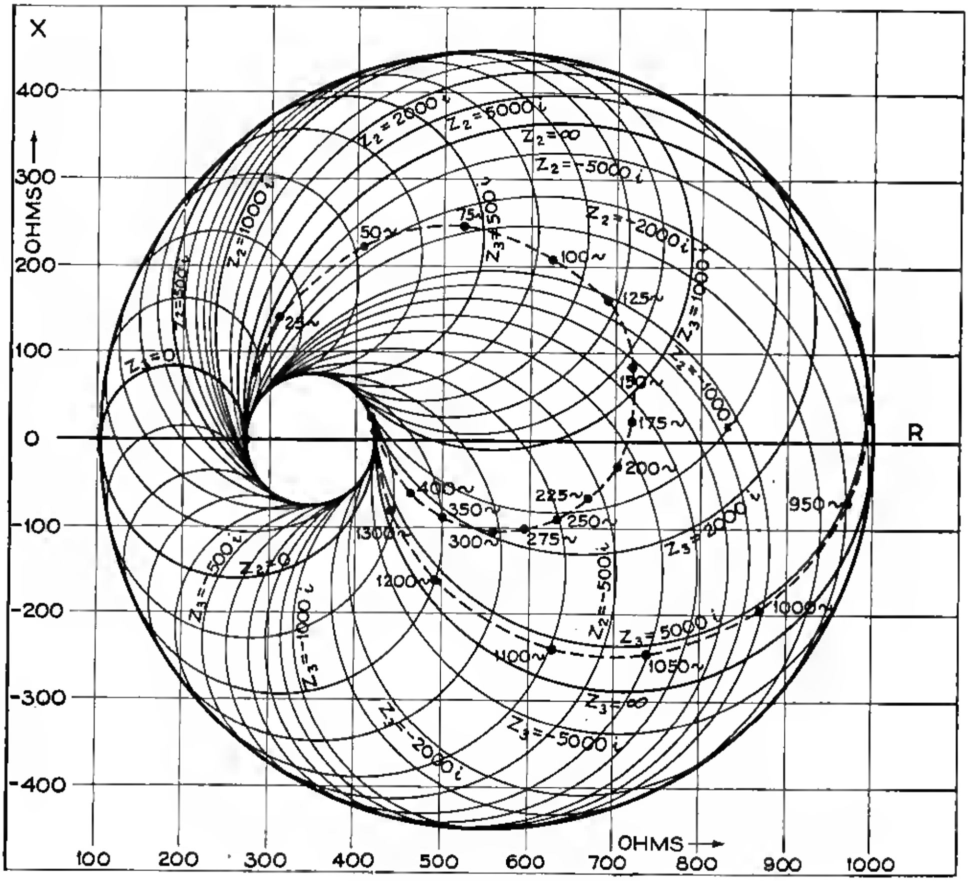
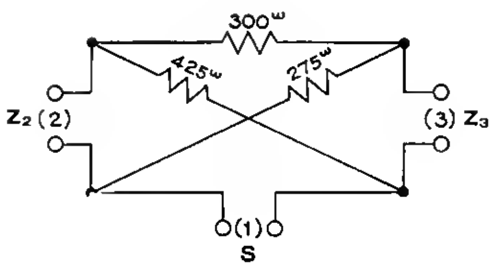


Fig. 6—Sheet II



The numbering of the sheets is, of course, arbitrary. If the upper half of the $Z_2=0$ circle is put on Sheet I, the arcs of the other critical circles are determined as follows:[6]

Circle	On Sheet I	On Sheet II
$Z_2=0$	Upper half	Lower half
$Z_2=\infty$	Lower half	Upper half
$Z_3=0$	Lower half	Upper half
$Z_3=\infty$	Upper half	Lower half

Each sheet, then, is divided into four sub-regions, indicated on Fig. 5 by the signs of the reactances for which S is within them. When Z_2 and Z_3 are composed of single elements the sub-regions in which S falls at any frequency are as follows:

Z_2	Z_3	Sub-Region	At Frequency Zero	At Frequency Infinity
Inductance	Inductance	$(+,+)$	$S=R_a$	$S=R_d$
Inductance	Capacity	$(+,-)$	R_c	R_b
Capacity	Inductance	$(-,+)$	R_b	R_c
Capacity	Capacity	$(-,-)$	R_d	R_a

The course of S over the complete frequency range may be shown by a curve through the appropriate intersections of the Z_2-constant and Z_3-constant circles, as in the following example.

The impedance region for a particular bridge network is illustrated in the two sheets of Fig. 6. The arcs of Z_2-constant and Z_3-constant circles in each sheet form a curvilinear grid superposed on the R,X grid of the complex plane. For example, if $Z_2=200i$ and $Z_3=900i$, the value of S is read from Sheet I as $327+i291$, and S has this value irrespective of the structure of Z_2 and Z_3.

An impedance curve (dashed) is shown in Fig. 6 representing the variation of S with frequency when Z_2 is the doubly-resonant reactance

[6] When the sheets are numbered in this way, the point S falls on Sheet I or Sheet II according to the following table, in which k_1 and k_2 are the critical values for the product and quotient of Z_2 and Z_3, respectively, given in Footnote 5:

(Z_2, Z_3)	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
On Sheet I, if	$Z_2/Z_3<k_2$	$Z_2Z_3>k_1$	$Z_2Z_3<k_1$	$Z_2/Z_3>k_2$
On Sheet II, if	$Z_2/Z_3>k_2$	$Z_2Z_3<k_1$	$Z_2Z_3>k_1$	$Z_2/Z_3<k_2$

For the network of Fig. 6, $k_1=116{,}875$ and $k_2=0.972111$.

used in Fig. 3b and Z_3 is an inductance of 1.0 henry. This impedance curve has four parts, two in each sheet. It starts on the resistance axis at the intersection of the $Z_2=\infty$ and $Z_3=0$ circles. As the frequency increases from zero the first part of the curve is traced out in Sheet II. At 25 cycles the impedance is approximately $310+i140$. The reactance component has a maximum of about 250 ohms at about 70 cycles, the resistance component has a maximum of about 720 ohms at about 160 cycles, the reactance component has a minimum of about -110 ohms at about 300 cycles, and finally at about 480 cycles the curve reaches the inner boundary, whereupon it changes to Sheet I. It remains in Sheet I up to a frequency of about 910 cycles, the resistance component having a minimum and the reactance component a maximum, which may be read from the diagram. The impedance between 910 cycles and approximately 1,390 cycles lies on Sheet II, and from 1,390 cycles to infinite frequency on Sheet I. The resistance component has a total of three maxima and three minima, and the reactance component three maxima and two minima, following the cyclical order: R-minimum, X-maximum, R-maximum, X-minimum.

An interesting exercise is to observe the effect on the impedance curve of changing the value of the inductance Z_3. The curve intersects the Z_2-constant circles at the same frequencies in each case, but the points of intersection are moved in a clockwise or counterclockwise sense as Z_3 is increased or decreased. With each such change parts of the impedance curve disappear from one sheet and reappear on the other. For instance, with a decrease of the inductance Z_3 the first loop of the impedance curve on Sheet II shrinks, and with sufficient decrease in inductance may become too small to plot, although it does not disappear entirely.

It is evident that if Z_2 and Z_3 are formed of reactance networks of greater complication the impedance curve may be very involved. But no matter how tortuous its path, it is restricted to the impedance region, that is, to the ring-shaped region between the non-intersecting boundary circles determined by the resistance network alone.

My thanks are due to Dr. George A. Campbell for his stimulating, continued interest, and to Mr. R. M. Foster for suggestions on every phase of this work.